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Intermixture of extended edge and localized bulk energy levels in macroscopic Hall systems

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Abstract

We study the spectrum of a random Schrödinger operator for an electron subjected to a magnetic field in a finite but macroscopic two-dimensional system of linear dimensions equal to L. The y direction is periodic and in the x direction the electron is confined by two smooth increasing boundary potentials. The eigenvalues of the Hamiltonian are classified according to their associated quantum mechanical diamagnetic current in the y direction. Here we look at an interval of energies inside the first Landau band of the random operator for the infinite plane. In this energy interval, with large probability, there exist $\mathcal{O}(L)$ eigenvalues with positive or negative currents of $\mathcal{O}(1)$. Between each of these there exist $\mathcal{O}(L^2)$ eigenvalues with infinitesimal current $\mathcal{O}\big(\mathrm{e}^{-\gamma B(\log L)^2}\big)$. We explain the relevance of this analysis of boundary diamagnetic currents to the integer quantum Hall effect.

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1. Introduction

In this paper we are concerned with the boundary currents in the integer quantum Hall effect that occurs in disordered electronic systems subject to a uniform magnetic field and confined in a two-dimensional interface of a heterojunction [1]. It was recognized by Halperin that boundary diamagnetic equilibrium currents play an important role in understanding the transport properties of such systems [2]. However, it was later realized that there is an intimate connection between these boundary currents and the topological properties of the state in the bulk [3, 4]. Here we will study only diamagnetic currents due to the boundaries, and not those produced by the adiabatic switching of an external infinitesimal electric field (as in linear response theory) which may exist in the bulk. Many features of the integral quantum Hall effect can be described in the framework of one-particle random magnetic Schrödinger operators and therefore it is important to understand their spectral properties for finite but macroscopic samples with boundaries. This problem has been approached recently for geometries where only one boundary is present and the operator is defined in a semi-infinite region [5–7].

Here we will take a finite system: our geometry is that of a cylinder of length and circumference both equal to L. There are two boundaries at $x = \pm\frac{L}{2}$ modelled by two smooth confining potentials $U_\ell(x)$ (ℓ for left) and $U_r(x)$ (r for right), and we take periodic boundary conditions in the y direction. These potentials vanish for $-\frac{L}{2} \leqslant x \leqslant \frac{L}{2}$ and grow fast enough for $|x| \geqslant \frac{L}{2}$. The Hamiltonian is of the form

$$H_\omega = H_0 + V_\omega + U_\ell + U_r \tag{1.1}$$

where H_0 is the pure Landau Hamiltonian for a uniform field of strength B and V_ω is a suitable weak random potential produced by impurities with $\sup|V_\omega(x, y)| = V_0 \ll B$ (see section 2 for precise assumptions). Before explaining our results it is useful to describe what is known about the infinite and semi-infinite cases.

In the case of the infinite plane $\mathbb{R}^2$ for the Hamiltonian $H_0 + V_\omega$ the spectrum forms 'Landau bands' contained in $\bigcup_{\nu\geqslant 0}\left[\left(\nu + \frac{1}{2}\right)B - V_0, \left(\nu + \frac{1}{2}\right)B + V_0\right]$. It is proved that the band tails have pure point spectrum corresponding to exponentially localized wavefunctions [8–12]. There are no rigorous results for energies at the band centres, except for a special model where the impurities are point scatterers [13, 14]. As first shown in [15] these spectral properties of random Schrödinger operators imply that the Hall conductivity—given by the Kubo formula—considered as a function of the filling factor (ratio of electron number and number of flux quanta) has quantized plateaux at values equal to $\nu e^2/h$, where ν is the number of filled Landau levels. The presence of the plateaux is a manifestation of Anderson localization while the quantization has a topological origin. The latter was first discovered in particular situations [16], and it has been proved for more general models using non-commutative geometry [17] and the index of Fredholm operators [18] (see [19] for a review).

In a semi-infinite system where the particle is confined in a half plane with Hamiltonian $H_0 + V_\omega + U_\ell$ (here (x, y) belongs to $\mathbb{R}^2$) the spectrum includes all energies in $\left[\frac{B}{2}, +\infty\right[$. The lower edge of the spectrum is between $\frac{B}{2} - V_0$ and $\frac{B}{2}$ and in its vicinity the spectrum is pure point (this follows from the techniques in [11]). For energies in intervals inside the gaps of the bulk Hamiltonian $H_0 + V_\omega$ the situation is completely different. One can show that the average velocity $(\psi, v_y\psi)$ in the y direction of an assumed eigenstate ψ does not vanish, but since the velocity v_y is the commutator between y and the Hamiltonian, this implies that the eigenstate cannot exist, and that therefore the spectrum is purely continuous [5, 20]. In fact, Mourre theory has been suitably applied to prove that the spectrum is purely absolutely continuous [6, 7]. These works put on a rigorous basis the expectation that, because of the chiral nature of the boundary currents, the states remain extended in the y direction even in the presence of disorder [2]. The same sort of analysis shows that if the y direction is made periodic of length L, the same energy intervals have discrete eigenstates which carry a current that is $\mathcal{O}(1)$—say positive—with respect to L [6]. Furthermore one can show that the eigenvalue spacing is of order $\mathcal{O}(L^{-1})$ [21].

The nature of the spectrum for a semi-infinite system for intervals inside the Landau bands of the bulk Hamiltonian $\bigcup_{\nu\geqslant 0}\left[\left(\nu + \frac{1}{2}\right)B - V_0, \left(\nu + \frac{1}{2}\right)B + V_0\right]$ has not yet been elucidated.

For the finite system on a cylinder with two boundaries the spectrum consists of finitely degenerate isolated eigenvalues. In [22], the results of [5, 6] for energy intervals inside the gaps of the bulk Hamiltonian are extended to the present two boundary system. The eigenvalues can be classified into two sets distinguished by the sign of their associated current[1]. These currents are uniformly positive or uniformly negative with respect to L. For this result to hold it is important to take the circumference and the length of the cylinder both of the order L.

[1] In principle the physical current is $L^{-1}(\psi, v_y\psi)$, but here we will call the current the average velocity $(\psi, v_y\psi)$.

In the present work, we study the currents of the eigenstates for eigenvalues in the interval $\Delta_\varepsilon = \left]\frac{B}{2}+\varepsilon, \frac{B}{2}+V_0\right[$ where ε is a small positive number independent of L. We limit ourselves to the first band to keep the discussion simpler. The content of our main result (theorem 1) is the following. Given ε, for L large enough there is an ensemble of realizations of the random potential with probability $1-\mathcal{O}(L^{-s})$ for which the eigenvalues of H_ω can be classified into three sets that we call Σ_ℓ, Σ_r and Σ_b. The eigenstates of Σ_r (resp. Σ_l) have uniformly positive (resp. negative) currents with respect to L, while those of Σ_b have a current of the order of $\mathcal{O}\big(\mathrm{e}^{-\gamma B(\log L)^2}\big)$. The number of eigenvalues in Σ_α ($\alpha = \ell, r$) is $\mathcal{O}(L)$ while that in Σ_b is $\mathcal{O}(L^2)$. This classification of eigenvalues leads to a well-defined notion of extended edge and localized bulk states. The edge states are those which belong to Σ_α ($\alpha = \ell, r$) and are extended in the sense that they have a current of order $\mathcal{O}(1)$. The bulk states are those which belong to Σ_b and are localized in the sense that their current is infinitesimal. The energy levels of the extended and localized states are *intermixed* in the same energy interval. See also [23] for a short review on spectral properties of systems defined on a cylinder.

Let us explain the mechanism that is at work. When the random potential is removed $V_\omega = 0$ in (1.1) the eigenstates with energies away from $\frac{B}{2}$ are extended in the y direction and localized in the x direction at a finite distance from the boundaries. Their energies form a sequence of 'edge levels' and have a spacing of the order of $\mathcal{O}(L^{-1})$. When the potential of one impurity is added to H_0 it typically creates a localized bound state with energy between the Landau levels. Suppose now that (i) a coupling constant in the impurity potential is *fine tuned* as a function of L so that the energy of the impurity level stays at a distance greater than L^{-p} from the edge levels, (ii) the position of the impurity is at a distance D from the boundaries. Then the mixing between the localized bound state and the extended edge states is controlled in second-order perturbation theory by the parameter $L^p\,\mathrm{e}^{-cBD^2}$. Therefore, one expects that bound states of impurities that have $D \gg (\log L)^{1/2}$ are basically unperturbed and have an infinitesimal current. On the other hand bound states coming from impurities with $D \ll (\log L)^{1/2}$ will mix with edge states. Note that *even for impurities with* $D \gg (\log L)^{1/2}$ *the coupling constant* (equivalently the impurity level) *has to be fine tuned* as a function of L. Indeed, for a coupling constant with a fixed value the energy of the impurity level is independent of L, and surely for L large enough the energy difference between the impurity and the edge levels becomes much smaller than $\mathcal{O}\big(\mathrm{e}^{-cBD^2}\big)$. Remarkably for a random potential, the absence of resonance is automatically achieved with large probability and no fine tuning is needed: this is why localized bulk states survive. We have analysed this mechanism rigorously for a model (see also [2]) where there are no impurities in a layer of thickness $(\log L)$ along the boundary. Then the edge levels are basically non-random and the typical spacing between current-carrying eigenvalues is easily controlled. Of course it is desirable to allow for impurities close to the boundary but then the edge levels become random and some further analysis is needed. However, we expect that the same basic mechanism operates because the typical spacing between edge levels should still be $\mathcal{O}(L^{-1})$. In connection with the discussion above we mention that for a semi-infinite system the bound state of an impurity at any fixed distance from the boundary turns into a resonance. A similar situation has been analysed in [24].

We note that the spectral region close to $\frac{B}{2}$ that is left out in our theorem is precisely that where resonances between edge and bulk states may occur because edge states become very dense. It is not clear what is the connection with the divergence of the localization length of the infinite system at the band centre.

In the present work we have shown that in quantum Hall samples there exist well-defined notions of extended edge states (current of $\mathcal{O}(1)$) and localized bulk states (infinitesimal current). Instead of classifying the energy levels according to their current one could try to use level statistics. We expect that the localized bulk states have Poissonian statistics whereas the extended edge states should display a level repulsion. In fact such a strong form of level repulsion is proved in [21] for energies in the gap of the bulk Hamiltonian where only extended edge states exist. It is interesting to observe that in the present situation both kinds of states have *intermixed* energy levels. In the usual Schrödinger operators (e.g. the Anderson model on a 3D cubic lattice) it is accepted (but not proved) that they are separated by a well-defined *mobilty edge* (results in this direction have been obtained recently [25] under a suitable hypothesis). The states at the band edge are localized in the sense that the spectrum is dense pure point for the infinite lattice and has Poisson statistics for the finite system [26]. At the band centre the states are believed to be extended in the sense that the spectrum is absolutely continuous for the infinite lattice and has the statistics of the Gaussian orthogonal ensemble for the finite lattice.

Other ways of formulating the notion of edge states have been proposed in different contexts. In [27] the authors consider a clean system with a novel kind of chiral boundary conditions. The Hilbert space then separates into two parts corresponding to edge and bulk states. The bulk states have exactly the Landau energy and the edge states a linear dispersion relation; the distinction between them being sharp because of the special nature of boundary conditions. It would be interesting to extend this definition to disordered systems. Recently in [28] (see also [29]) another approach has been used in the context of magnetic billards. The authors study a magnetic billiard with mixed boundary conditions with mixing parameter Λ interpolating between Dirichlet and Neumann boundary conditions. They look at the sensibility of the eigenstates and eigenvalues under the variation of Λ and define in this way an edge state as a state that depends strongly on Λ. Let us note that our notion of edge state as well as the other ones all share the feature that an edge state carries a substantial current.

The characterization of the spectrum of (1.1) proposed here also has a direct relevance to the Hall conductivity of the many-electron (non-interacting) system. In the formulation advocated by Halperin [2] the Hall conductivity is computed as the ratio of the net equilibrium current and the difference of chemical potentials between the two edges. Consider the many-fermion state $\Psi(\mu_\ell, \mu_r, E_F)$ obtained by filling the levels of H_ω (one particle per state) in $\Sigma_\ell \cap \left[\frac{B}{2} + \varepsilon, \mu_\ell\right]$, $\Sigma_r \cap \left[\frac{B}{2} + \varepsilon, \mu_r\right]$ and $\Sigma_b \cap \left[\frac{B}{2} + \varepsilon, E_F\right]$ with $\frac{B}{2} + \varepsilon < \mu_\ell < E_F < \mu_r < \frac{B}{2} + V_0$. The total current $I(\mu_\ell, \mu_r, E_F)$ of this state—a stationary state of the many particle Hamiltonian—is given by the sum of the individual physical currents of the filled levels (given by $L^{-1}(\psi, v_y\psi)$). From the estimates (2.16) and (2.18) in theorem 1

$$\sum_k J_k^\ell + \sum_k J_k^r + \sum_\beta J_\beta = \sum_k J_{0k}^\ell + \sum_k J_{0k}^r + \mathcal{O}\big(\mathrm{e}^{-(\log L)^2} L^2\big) \tag{1.2}$$

and from (2.10) we get

$$\frac{1}{L}\sum_k J_{0k}^r = \frac{1}{2\pi}\int_{\frac{B}{2}+\varepsilon}^{\mu_r} \mathrm{d}E + \mathcal{O}(L^{-1}) \tag{1.3}$$

$$\frac{1}{L}\sum_k J_{0k}^\ell = \frac{1}{2\pi}\int_{\mu_\ell}^{\frac{B}{2}+\varepsilon} \mathrm{d}E + \mathcal{O}(L^{-1}). \tag{1.4}$$

It follows that to leading order

$$I(\mu_\ell, \mu_r, E_F) \simeq \frac{1}{2\pi}(\mu_r - \mu_\ell). \tag{1.5}$$

In (1.5) the Hall conductance is equal to 1 (this is because we have considered only the first band). When μ_ℓ and μ_r vary the density of particles in the state, $\Psi(\mu_\ell, \mu_r, E_F)$ does not change since the number of levels in Σ_α $(\alpha = \ell, r)$ is of order $\mathcal{O}(L)$. However, if E_F is increased the particle density (and thus the filling factor) increases since the number of levels in Σ_b is of order $\mathcal{O}(L^2)$, but the Hall conductance does not change and hence has a plateau. In other words the edge states contribute to the Hall conductance but not to the density of states of the sample in the thermodynamic limit.

In a more complete theory one should also take into account currents possibly flowing in the bulk due to the adiabatic switching of an external electric field, an issue that is beyond the scope of the present analysis. A related problem is the relationship between the conductance in the present picture, defined through (1.5), and that using the Kubo formula (see [30–32]).

The precise definition of the model and the statement of the main result (theorem 1) are the subject of the next section.

2. The structure of the spectrum

We consider the family of random Hamiltonians (1.1) acting on the Hilbert space $L^2\big(\mathbb{R} \times \big[-\frac{L}{2}, \frac{L}{2}\big]\big)$ with periodic boundary conditions along y, $\psi\big(x, -\frac{L}{2}\big) = \psi\big(x, \frac{L}{2}\big)$. In the Landau gauge the kinetic term of (1.1) is

$$H_0 = \tfrac{1}{2} p_x^2 + \tfrac{1}{2}(p_y - Bx)^2 \tag{2.1}$$

and has infinitely degenerate Landau levels $\sigma(H_0) = \big\{\big(\nu + \frac{1}{2}\big)B;\ \nu \in \mathbb{N}\big\}$. We will make extensive use of explicit pointwise bounds, proved in appendix A, on the integral kernel of the resolvent $R_0(z) = (z - H_0)^{-1}$ with periodic boundary conditions along y.

The confining potentials modelling the two edges at $x = -\frac{L}{2}$ and $x = \frac{L}{2}$ are assumed to be strictly monotonic, differentiable and such that

$$c_1 \left| x + \frac{L}{2} \right|^{m_1} \leqslant U_\ell(x) \leqslant c_2 \left| x + \frac{L}{2} \right|^{m_2} \qquad \text{for} \quad x \leqslant -\frac{L}{2} \tag{2.2}$$

$$c_1 \left| x - \frac{L}{2} \right|^{m_1} \leqslant U_r(x) \leqslant c_2 \left| x - \frac{L}{2} \right|^{m_2} \qquad \text{for} \quad x \geqslant \frac{L}{2} \tag{2.3}$$

for some constants $0 < c_1 < c_2$ and $2 \leqslant m_1 < m_2 < \infty$. Recall that $U_\ell(x) = 0$ for $x \geqslant -\frac{L}{2}$ and $U_r(x) = 0$ for $x \leqslant \frac{L}{2}$. We could allow steeper confinements but the present polynomial conditions turn out to be technically convenient.

We assume that each impurity is the source of a local potential $V \in C^2$, $0 \leqslant V(x, y) \leqslant V_0 < \infty$, $\operatorname{supp} V \subset \mathbb{B}\big(0, \frac{1}{4}\big)$, and that impurities are located at the sites of a finite lattice $\Lambda = \big\{(n, m) \in \mathbb{Z}^2;\ n \in \big[-\frac{L}{2} + \log L, \frac{L}{2} - \log L\big], m \in \big[-\frac{L}{2}, \frac{L}{2}\big]\big\}$. The random potential V_ω has the form

$$V_\omega(x, y) = \sum_{(n,m)\in\Lambda} X_{n,m}(\omega) V(x - n, y - m) \tag{2.4}$$

where the coupling constants $X_{n,m}$ are i.i.d. random variables with common density $h \in C^2([-1, 1])$ that satisfies $\|h\|_\infty < \infty$, $\operatorname{supp} h = [-1, 1]$. We will denote by $\mathbb{P}_\Lambda$ the product measure defined on the set of all possible realizations $\omega \in \Omega_\Lambda = [-1, 1]^\Lambda$. Clearly for any realization we have $|V_\omega(x, y)| \leqslant V_0$. Furthermore, it will be assumed that the random potential is weak in the sense that $4V_0 < B$.

We will think of our system as being constituted of three pieces corresponding to the *bulk system* with the random Hamiltonian

$$H_b = H_0 + V_\omega \tag{2.5}$$

and the *left* and *right edge systems* with non-random Hamiltonians

$$H_\alpha = H_0 + U_\alpha \qquad \alpha = \ell, r. \tag{2.6}$$

All the Hamiltonians considered above have periodic boundary conditions along the y direction and are essentially self-adjoint on $C_0^\infty\big(\mathbb{R}\times\big[-\frac{L}{2},\frac{L}{2}\big]\big)$. For each realization ω and size L the spectrum $\sigma(H_\omega)$ of (1.1) (it depends on L) consists of isolated eigenvalues of finite multiplicity. In order to state our main result characterizing these eigenvalues we first have to describe the spectra of (2.5) and (2.6).

Let us begin with the edge Hamiltonians (2.6). Here we state their properties without proofs and refer the reader to [5, 20] for more details. Since the edge Hamiltonians H_α commute with p_y, they decomposed into a direct sum:

$$H_\alpha = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}}^{\oplus} H_\alpha(k) = \sum_{k\in\frac{2\pi}{L}\mathbb{Z}}^{\oplus}\left[\frac{1}{2}p_x^2 + \frac{1}{2}(k-Bx)^2 + U_\alpha\right]. \tag{2.7}$$

For each k the one-dimensional Hamiltonian $H_\alpha(k)$ has a compact resolvent, thus it has discrete eigenvalues and by standard arguments one can show that they are not degenerate. If the y direction would be infinitely extended, k would vary over the real axis and the eigenvalues of $H_\alpha(k)$ would form spectral branches $\varepsilon_\nu^\alpha(\hat{k})$, $\hat{k}\in\mathbb{R}$ labelled by the Landau level index ν. These spectral branches are strictly monotone, entire functions with the properties $\varepsilon_\nu^\ell(-\infty)=+\infty$, $\varepsilon_\nu^\ell(+\infty)=\big(\nu+\frac{1}{2}\big)B$ and $\varepsilon_\nu^r(-\infty)=\big(\nu+\frac{1}{2}\big)B$, $\varepsilon_\nu^r(+\infty)=+\infty$. Here because of the periodic boundary conditions the set of k values is discrete so that the spectrum of H_α

$$\sigma(H_\alpha)=\left\{E_{\nu k}^\alpha;\ \nu\in\mathbb{N}, k\in\frac{2\pi}{L}\mathbb{Z}\right\} \tag{2.8}$$

consists of isolated points on the spectral branches $E_{\nu k}^\alpha=\varepsilon_\nu^\alpha(k)$, $k\in\frac{2\pi}{L}\mathbb{Z}$. The corresponding eigenfunctions $\psi_{\nu k}^\alpha$ have the form

$$\psi_{\nu k}^\alpha(x,y)=\frac{1}{\sqrt{L}}\,\mathrm{e}^{\mathrm{i}ky}\varphi_{\nu k}^\alpha(x) \tag{2.9}$$

with $\varphi_{\nu k}^\alpha$ the normalized eigenfunctions of the one-dimensional Hamiltonian $H_\alpha(k)$. By definition, the current of the state $\psi_{\nu k}^\alpha$ in the y direction is given by the expectation value of the velocity $v_y=p_y-Bx$,

$$J_{\nu k}^\alpha=\big(\psi_{\nu k}^\alpha, v_y\psi_{\nu k}^\alpha\big)=\int_{\mathbb{R}}\left|\varphi_{\nu k}^\alpha(x)\right|^2(k-Bx)\,\mathrm{d}x=\partial_{\hat{k}}\varepsilon_\nu^\alpha(\hat{k})\Big|_{\hat{k}=\frac{2\pi m}{L}} \tag{2.10}$$

where the last equality follows from the Feynman–Hellman theorem. From (2.10) we note that for any $\varepsilon>0$, one can find $j(\varepsilon)>0$ and $L(\varepsilon)$ such that for $L>L(\varepsilon)$ the states of the two branches $\nu=0,\alpha=\ell,r$ with energies $E_{0k}^\alpha\geqslant\frac{1}{2}B+\varepsilon$ satisfy

$$J_{0k}^\ell\leqslant -j(\varepsilon)<0 \qquad J_{0k}^r\geqslant j(\varepsilon)>0. \tag{2.11}$$

In other words the eigenstates of the edge Hamiltonians carry an appreciable current. The spacing of two consecutive eigenvalues greater than $\frac{1}{2}B+\varepsilon$ satisfies

$$\left|E_{0\frac{2\pi(m+1)}{L}}^\alpha - E_{0\frac{2\pi m}{L}}^\alpha\right| > \frac{j(\varepsilon)}{L} \qquad \alpha=\ell,r. \tag{2.12}$$

Note that these observations extend to other branches but $j(\varepsilon)$ and $L(\varepsilon)$ are not uniform with respect to the index ν. In the rest of the paper we limit ourselves to $\nu=0$ for simplicity. On the other hand the spacing between the energies of $\sigma(H_\ell)$ and $\sigma(H_r)$ is *a priori* arbitrary. We assume that the confining potentials U_ℓ and U_r are such that the following hypothesis is fulfilled.

Hypothesis 1. *Fix any* $\varepsilon > 0$ *and let* $\Delta_\varepsilon = \left[\frac{1}{2}B + \varepsilon, \frac{1}{2}B + V_0\right]$. *There exist* $L(\varepsilon)$ *and* $d(\varepsilon) > 0$ *such that for all* $L > L(\varepsilon)$

$$\operatorname{dist}(\sigma(H_\ell) \cap \Delta_\varepsilon, \sigma(H_r) \cap \Delta_\varepsilon) \geqslant \frac{d(\varepsilon)}{L}. \tag{2.13}$$

This hypothesis is important because a minimal amount of non-degeneracy between the spectra of the two edge systems is needed in order to control backscattering effects induced by the random potential. Indeed, in a system with two boundaries backscattering favours localization and has a tendency to destroy currents. This hypothesis can easily be realized by taking non-symmetric confining potentials U_ℓ and U_r. In a more realistic model with impurities close to the edges one expects that it is automatically satisfied with a large probability.

Now we describe the spectral properties of the bulk random Hamiltonian (2.5). From the bound (A.5) on the kernel of $R_0(z)$ and the fact that V_ω is bounded with compact support we can see that V_ω is relatively compact w.r.t. H_0, thus $\sigma_{ess}(H_b) = \left\{\left(\nu + \frac{1}{2}\right)B; \nu \in \mathbb{N}\right\}$. Since $|V_\omega(x, y)| \leqslant V_0 < B$ the eigenvalues E_β^b of H_b are contained in Landau bands $\bigcup_{\nu\geqslant 0}\left[\left(\nu + \frac{1}{2}\right)B - V_0, \left(\nu + \frac{1}{2}\right)B + V_0\right]$. We will assume

Hypothesis 2. *Fix any* $\varepsilon > 0$. *There exist* $\mu(\varepsilon)$ *a strictly positive constant and* $L(\varepsilon)$ *such that for all* $L > L(\varepsilon)$ *one can find a set of realizations of the random potential* Ω'_Λ *with* $\mathbb{P}_\Lambda(\Omega'_\Lambda) \geqslant 1 - L^{-\theta}$, $\theta > 0$, *with the property that if* $\omega \in \Omega'_\Lambda$ *the eigenstates corresponding to* $E_\beta^b \in \sigma(H_b) \cap \Delta_\varepsilon$ *satisfy*

$$\left|\psi_\beta^b(x, \bar{y}_\beta)\right| \leqslant \mathrm{e}^{-\mu(\varepsilon)L} \qquad \left|\partial_y \psi_\beta^b(x, \bar{y}_\beta)\right| \leqslant \mathrm{e}^{-\mu(\varepsilon)L} \tag{2.14}$$

for some $\bar{y}_\beta$ *depending on* ω *and* L.

Since V_ω is random we expect that wavefunctions with energies in Δ_ε (not too close to the Landau levels where the localization length diverges) are exponentially localized on a scale $\mathcal{O}(1)$ with respect to L. Inequalities (2.14) are a weaker version of this statement and have been checked for the special case where the random potential is a sum of rank-one perturbations [33] using the methods of Aizenman and Molchanov [34] (see for example [14] where the case of point impurities is treated by these methods). Presumably one could adapt existing techniques for multiplicative potentials to our geometry, to prove hypothesis $(H2)$ at least for energies close to the band tail $\frac{B}{2} + V_0$. One also expects that $\mu(\varepsilon) \to 0$ as $\varepsilon \to 0$. The main physical consequence of $(H2)$ (as shown in section 5) is that a state satisfying (2.14) does not carry any appreciable current (contrary to the eigenstates of H_α) in the sense that $J_\beta^b = \left(\psi_\beta^b, v_y\psi_\beta^b\right) = \mathcal{O}(\mathrm{e}^{-\mu(\varepsilon)L})$.

We now state our main result.

Theorem 1. *Fix* $\varepsilon > 0$ *and assume that* $(H1)$ *and* $(H2)$ *are fulfilled. Assume* $B > 4V_0$. *Let* $p \geqslant 7$ *and* $s = \min(\theta, p - 6)$. *Then there exists a numerical constant* $\gamma > 0$ *and an* $L(\varepsilon, p, B, V_0)$ *such that for all* $L > L(\varepsilon, p, B, V_0)$ *one can find a set* $\hat{\Omega}_\Lambda$ *of realizations of the random potential with* $\mathbb{P}_\Lambda(\hat{\Omega}_\Lambda) \geqslant 1 - 3L^{-s}$ *such that for any* $\omega \in \hat{\Omega}_\Lambda$, $\sigma(H_\omega) \cap \Delta_\varepsilon$ *is the union of three sets* $\Sigma_\ell \cup \Sigma_b \cup \Sigma_r$, *each depending on* ω *and* L, *and characterized by the following properties:*

(a) $E_k^\alpha \in \Sigma_\alpha$ $(\alpha = \ell, r)$ *are a small perturbation of* $E_{0k}^\alpha \in \sigma(H_\alpha) \cap \Delta_\varepsilon$ *with*

$$\left|E_k^\alpha - E_{0k}^\alpha\right| \leqslant \mathrm{e}^{-\gamma B(\log L)^2} \qquad \alpha = \ell, r. \tag{2.15}$$

(b) For $E_k^\alpha \in \Sigma_\alpha$ *the current* J_k^α *of the associated eigenstate satisfies*

$$\left|J_k^\alpha - J_{0k}^\alpha\right| \leqslant \mathrm{e}^{-\gamma B(\log L)^2} \qquad \alpha = \ell, r. \tag{2.16}$$

(c) Σ_b *contains the same number of energy levels as* $\sigma(H_b) \cap \Delta_\varepsilon$ *and*

$$\mathrm{dist}(\Sigma_b, \Sigma_\alpha) \geqslant L^{-p+1} \qquad \alpha = \ell, r. \tag{2.17}$$

(d) *The current associated with each level* $E_\beta \in \Sigma_b$ *satisfies*

$$|J_\beta| \leqslant \mathrm{e}^{-\gamma B(\log L)^2}. \tag{2.18}$$

The proof of the theorem is organized as follows. In section 3 we set up a decoupling scheme by which we express the resolvent of H_ω as an approximate sum of those of the edge and bulk systems. Parts (a) and (c) of theorem 1 are proved in section 4. First we compute approximations for the spectral projections of H_ω in terms of the projectors $P\left(E_{0k}^\alpha\right)$ of H_α and $P_b(\bar{\Delta})$ of H_b (proposition 1). This is done for realizations of the disorder such that the levels of H_b are not 'too close' to those of H_α. We then show that these realizations are typical (have large probability) thanks to a Wegner estimate (proposition 2). Parts (b) and (d) are proved in section 5 by estimating currents in terms of norms of differences between projectors. The appendices contain some technical estimates.

3. Decoupling of the bulk and the edge systems

The resolvent $R(z) = (z - H_\omega)^{-1}$ can be expressed, up to a small term, as a sum of the resolvents of the bulk system $R_b(z) = (z - H_b)^{-1}$ and the two edge systems $R_\alpha(z) = (z - H_\alpha)^{-1}$ $(\alpha = \ell, r)$. Here this will be achieved by a *decoupling formula* developed in other contexts [35, 36]. We set $D = \log L$ and introduce the characteristic functions

$$\tilde{J}_\ell(x) = \chi_{]-\infty,-\frac{L}{2}+\frac{D}{2}]}(x) \qquad \tilde{J}_b(x) = \chi_{[-\frac{L}{2}+\frac{D}{2},\frac{L}{2}-\frac{D}{2}]}(x) \qquad \tilde{J}_r(x) = \chi_{[-\frac{L}{2}+\frac{D}{2},+\infty[}(x) \tag{3.1}$$

We will also use three bounded $C^\infty(\mathbb{R})$ functions $|J_i(x)| \leqslant 1$, $i \in \mathcal{I} \equiv \{\ell, b, r\}$, with bounded first and second derivatives $\sup_x \left|\partial_x^n J_i(x)\right| \leqslant 2$, $n = 1, 2$, and such that

$$J_\ell(x) = \begin{cases} 1 & \text{if} \quad x \leqslant -\frac{L}{2} + \frac{3D}{4} \\ 0 & \text{if} \quad x \geqslant -\frac{L}{2} + \frac{3D}{4} + 1 \end{cases} \qquad J_b(x) = \begin{cases} 1 & \text{if} \quad |x| \leqslant \frac{L}{2} - \frac{D}{4} \\ 0 & \text{if} \quad |x| \geqslant \frac{L}{2} - \frac{D}{4} + 1 \end{cases}$$
$$J_r(x) = \begin{cases} 1 & \text{if} \quad x \geqslant \frac{L}{2} - \frac{3D}{4} \\ 0 & \text{if} \quad x \leqslant \frac{L}{2} - \frac{3D}{4} - 1. \end{cases} \tag{3.2}$$

For $i \in \mathcal{I}$ we have $H_\omega J_i = H_i J_i$ thus

$$(z - H_\omega) \sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i = \sum_{i \in \mathcal{I}} (z - H_i) J_i R_i(z) \tilde{J}_i = 1 - \mathcal{K}(z) \tag{3.3}$$

where

$$\mathcal{K}(z) = \sum_{i \in \mathcal{I}} K_i(z) = \sum_{i \in \mathcal{I}} \frac{1}{2} \left[p_x^2, J_i\right] R_i(z) \tilde{J}_i. \tag{3.4}$$

To obtain the second equality one commutes $(z - H_i)$ and J_i and then uses the identity $\sum_{i \in \mathcal{I}} J_i \tilde{J}_i = \sum_{i \in \mathcal{I}} \tilde{J}_i = 1$. From (3.3) we deduce the decoupling formula

$$R(z) = \left(\sum_{i \in \mathcal{I}} J_i R_i(z) \tilde{J}_i\right) (1 - \mathcal{K}(z))^{-1}. \tag{3.5}$$

The main result of this section is an estimate of the operator norm of $\mathcal{K}(z)$. In particular it will assure $\|\mathcal{K}(z)\| < 1$.

Lemma 1. *Let* $\operatorname{Re} z \in \Delta_\varepsilon$ *such that* $\operatorname{dist}(z, \sigma(H_\ell) \cup \sigma(H_r) \cup \sigma(H_b)) \geqslant \mathrm{e}^{-\frac{B}{512}(\log L)^2}$. *Then for L large enough there exists a constant* $C(B, V_0) > 0$ *independent of L such that*

$$\|\mathcal{K}(z)\| \leqslant \varepsilon^{-1} C(B, V_0) L\, \mathrm{e}^{-\frac{B}{512}(\log L)^2}. \tag{3.6}$$

Proof. Computing the commutator in the definition of $K_i(z)$ and applying the second resolvent formula we have

$$\begin{aligned} K_i(z) &= -\tfrac{1}{2}\left(\partial_x^2 J_i\right) R_i(z)\tilde{J}_i - (\partial_x J_i)\partial_x R_i(z)\tilde{J}_i \\ &= -\tfrac{1}{2}\left(\partial_x^2 J_i\right) R_0(z)\tilde{J}_i - \tfrac{1}{2}\left(\partial_x^2 J_i\right) R_0(z) W_i R_i(z)\tilde{J}_i \\ &\quad - (\partial_x J_i)\partial_x R_0(z)\tilde{J}_i - (\partial_x J_i)\partial_x R_0(z) W_i R_i(z)\tilde{J}_i \end{aligned} \tag{3.7}$$

where we have set $W_\ell = U_\ell$, $W_b = V_\omega$ and $W_r = U_r$. From the triangle inequality and $\|R_i(z)\| = \operatorname{dist}(z, \sigma(H_i))^{-1}$ we obtain

$$\begin{aligned} \|K_i(z)\| &\leqslant \tfrac{1}{2}\left\|\left(\partial_x^2 J_i\right) R_0(z)\tilde{J}_i\right\| + \tfrac{1}{2}\left\|\left(\partial_x^2 J_i\right) R_0(z) W_i\right\| \operatorname{dist}(z, \sigma(H_i))^{-1} \\ &\quad + \|(\partial_x J_i)\, \partial_x R_0(z)\tilde{J}_i\| + \|(\partial_x J_i)\partial_x R_0(z) W_i\| \operatorname{dist}(z, \sigma(H_i))^{-1}. \end{aligned} \tag{3.8}$$

To estimate the operator norms on the right-hand side it is sufficient to bound them by the Hilbert–Schmidt norms $\|\cdot\|_2$. Using bounds (A.5) on the kernels of $\partial_x^n R_0(z)$ for $n = 0, 1$, and the properties of the functions J_i, $\tilde{J}_i$ we obtain

$$\begin{aligned} \left\|\left(\partial_x^{2-n} J_i\right) \partial_x^n R_0(z)\tilde{J}_i\right\|_2^2 &= \int_{\operatorname{supp}\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \left|\partial_x^{2-n} J_i(x)\right|^2 \int_{\operatorname{supp}\tilde{J}_i} \mathrm{d}\mathbf{x}' \left|\partial_x^n R_0(\mathbf{x}, \mathbf{x}'; z)\right|^2 \\ &\leqslant 4C_n^2(z, B) \int_{\operatorname{supp}\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \int_{\operatorname{supp}\tilde{J}_i} \mathrm{d}\mathbf{x}'\, \mathrm{e}^{-\frac{B}{4}(x-x')^2} \\ &\leqslant 4C_n^2(z, B)\, \mathrm{e}^{-\frac{B}{8}(\frac{D}{4}+1)^2} \int_{\operatorname{supp}\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \int_{\mathbb{R}\times[-L/2, L/2]} \mathrm{d}\mathbf{x}'\, \mathrm{e}^{-\frac{B}{8}(x-x')^2} \\ &\leqslant 16\sqrt{\frac{\pi}{B}} C_n^2(z, B) L^2\, \mathrm{e}^{-\frac{B}{128}D^2}. \end{aligned} \tag{3.9}$$

For the norms involving the potentials W_i we obtain in a similar way

$$\begin{aligned} \left\|\partial_x^{2-n} J_i \partial_x^n R_0(z) W_i\right\|_2^2 &= \int_{\operatorname{supp}\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \left|\partial_x^{2-n} J_i(x)\right|^2 \int_{\operatorname{supp} W_i} \mathrm{d}\mathbf{x}' \left|\partial_x^\alpha R_0(\mathbf{x}, \mathbf{x}'; z)\right|^2 |W_i(\mathbf{x}')|^2 \\ &\leqslant 4C_n^2(z, B)\, \mathrm{e}^{-\frac{B}{128}D^2} \int_{\operatorname{supp}\partial_x^{2-n} J_i} \mathrm{d}\mathbf{x} \int_{\operatorname{supp} W_i} \mathrm{d}\mathbf{x}'\, \mathrm{e}^{-\frac{B}{8}(x-x')^2} |W_i(\mathbf{x}')|^2. \end{aligned} \tag{3.10}$$

It is clear that since V_ω is bounded, and U_ℓ, U_r do not grow faster than polynomials, the double integral on the right-hand side of the last inequality is bounded above by L^2 times a constant depending only on B and V_0. From this result, (3.8), (3.9) and $\operatorname{dist}(z, \sigma(H_\ell) \cup \sigma(H_r) \cup \sigma(H_b)) \geqslant \mathrm{e}^{-\frac{B}{512}(\log L)^2}$ we obtain ($\tilde{C}(B, V_0)$ a constant independent of L)

$$\|K_i(z)\| \leqslant \tilde{C}(B, V_0)\varepsilon^{-1} L\, \mathrm{e}^{-\frac{B}{512}(\log L)^2} \tag{3.11}$$

where we used the expression for $C_n(z, B)$ in appendix A and the fact that $\operatorname{Re} z \in \Delta_\varepsilon$. □

4. Estimates of eigenprojectors of H_ω

In this section we use the decoupling formula (3.5) to give deterministic estimates for the difference between projectors of H_ω and H_b, H_ℓ and H_r. We then combine this information with a probabilistic estimate (Wegner estimate) to deduce that the spectrum of H_ω is the union of the three sets Σ_ℓ, Σ_r and Σ_b satisfying parts (a) and (c) of theorem 1.

Proposition 1. *Assume that* $(H1)$ *holds. Take* $p \geqslant 7$ *and any* $\mathrm{e}^{-\frac{B}{512}(\log L)^2} < \rho < \frac{d(\varepsilon)}{2}L^{-p}$. *For* $L > L(\varepsilon)$ *let* Ω''_Λ *be the set of realizations of the random potential such that for each* $\omega \in \Omega''_\Lambda$ $\operatorname{dist}\left(\sigma(H_b)\cap\Delta_\varepsilon, E^\alpha_{0k}\right) \geqslant d(\varepsilon)L^{-p}$ *for all* $E^\alpha_{0k}\in\Delta_\varepsilon$, $\alpha = \ell, r$. *Then*

(i) If $P\left(E^\alpha_{0k}\right)$ *is the eigenprojector of* H_α *associated with the eigenvalue* $E^\alpha_{0k}\in\Delta_\varepsilon$ *and* P^α_k *the eigenprojector of* H_ω *for the intervals* $I^\alpha_k = \left[E^\alpha_{0k}-\rho, E^\alpha_{0k}+\rho\right]$ *we have*

$$\left\|P^\alpha_k - P\left(E^\alpha_{0k}\right)\right\| \leqslant \varepsilon^{-1}C'(B,V_0)L\,\mathrm{e}^{-\frac{B}{512}(\log L)^2}. \tag{4.1}$$

(ii) Let $\bar\Delta\subset\Delta_\varepsilon$ *be an interval such that* $\operatorname{dist}(\bar\Delta,\sigma(H_\ell)\cup\sigma(H_r)) = \frac{d(\varepsilon)}{2}L^{-p}$. *If* $P_b(\bar\Delta)$ *is the eigenprojector of* H_b *for the interval* $\bar\Delta$ *and* $P(\bar\Delta)$ *the eigenprojector of* H_ω *for the interval* $\bar\Delta$ *we have*

$$\|P(\bar\Delta) - P_b(\bar\Delta)\| \leqslant \varepsilon^{-3}C'(B,V_0)L^p\,\mathrm{e}^{-\frac{B}{512}(\log L)^2}. \tag{4.2}$$

Proof. We start by proving (4.1) for $\alpha = r$. The case $\alpha = \ell$ is identical. From the decoupling formula we have

$$\begin{aligned} R(z) - R_r(z) = {} & \left(\sum_{i\in\mathcal{I}} J_i R_i(z)\tilde J_i\right)\left(\sum_{n=1}^{\infty}\mathcal{K}(z)^n\right) - (1-J_r)R_r(z) \\ & - J_rR_r(z)(1-\tilde J_r) + J_\ell R_\ell(z)\tilde J_\ell + J_bR_b(z)\tilde J_b. \end{aligned} \tag{4.3}$$

Let Γ be a circle of radius ρ in the complex plane, centred at E^r_{0k}. Because of $(H1)$ and $\operatorname{dist}\left(\sigma(H_b)\cap\Delta_\varepsilon, E^r_{0k}\right)\geqslant d(\varepsilon)L^{-p}$, $R_b(z)$ and $R_\ell(z)$ have no poles in Γ. Moreover the only pole of $R_r(z)$ is precisely E^r_{0k}. Thus integrating (4.3) along the circle Γ

$$\begin{aligned} P^r_k - P\left(E^r_{0k}\right) = {} & \frac{1}{2\pi\mathrm{i}}\oint_\Gamma\left(\sum_{i\in\mathcal{I}} J_iR_i(z)\tilde J_i\right)\sum_{n=1}^{\infty}\mathcal{K}(z)^n\,\mathrm{d}z \\ & - (1-J_r)P\left(E^r_{0k}\right) - J_rP\left(E^r_{0k}\right)(1-\tilde J_r). \end{aligned} \tag{4.4}$$

We proceed to estimate the norms of the three contributions on the right-hand side of (4.4). The norm of the first term is smaller than

$$\rho\left(\sum_{i\in\mathcal{I}}\sup_{z\in\Gamma}\|R_i(z)\|\right)\frac{\sup_{z\in\Gamma}\|\mathcal{K}(z)\|}{1-\sup_{z\in\Gamma}\|\mathcal{K}(z)\|} \leqslant 6\varepsilon^{-1}C(B,V_0)L\,\mathrm{e}^{-\frac{B}{512}(\log L)^2}. \tag{4.5}$$

Indeed, for $i = r$ we have $\sup_{z\in\Gamma}\|R_r(z)\| = \rho^{-1}$ by construction. For $i = \ell, b$ we have $\sup_{z\in\Gamma}\|R_i(z)\| < \frac{2}{d(\varepsilon)}L^p$. Since $\rho < \frac{d(\varepsilon)}{2}L^{-p}$ we note that in all three cases $(i\in\mathcal{I})$ $\rho\sup_{z\in\Gamma}\|R_i(z)\|\leqslant 1$. Furthermore, since $\rho > \mathrm{e}^{-\frac{B}{512}(\log L)^2}$, using lemma 1 we get (4.5). To estimate the second term in (4.4) we note that by the second resolvent formula

$$\frac{P\left(E^r_{0k}\right)}{\left(z-E^r_{0k}\right)} = (z-H_0)^{-1}P_r\left(E^r_{0k}\right) + (z-H_0)^{-1}U_r\frac{P\left(E^r_{0k}\right)}{\left(z-E^r_{0k}\right)}. \tag{4.6}$$

Integrating (4.6) along Γ we obtain the identity

$$P\left(E^r_{0k}\right) = \left(E^r_{0k}-H_0\right)^{-1}U_rP\left(E^r_{0k}\right) \tag{4.7}$$

this implies

$$\begin{aligned} \left\|(1-J_r)P\left(E^r_{0k}\right)\right\| & \leqslant \left\|(1-J_r)R_0\left(E^r_{0k}\right)U_r\right\| \leqslant \left\|(1-J_r)R_0\left(E^r_{0k}\right)U_r\right\|_2 \\ & = \left\{\int\mathrm{d}\mathbf{x}|1-J_r(x)|^2\int\mathrm{d}\mathbf{x}'\left|R_0\left(\mathbf{x},\mathbf{x}';E^r_{0k}\right)U_r(x')\right|^2\right\}^{1/2} \end{aligned} \tag{4.8}$$

since the distance (in the x direction) between the supports of $(1 - J_r)$ and U_r is greater than $\frac{D}{2} + 1$ we can proceed in a similar way as in the estimate of (3.10) to obtain

$$\left\|(1 - J_r)P\left(E_{0k}^r\right)\right\| \leqslant \varepsilon^{-1}\bar{C}(B)L\,\mathrm{e}^{-\frac{B}{64}(\log L)^2} \tag{4.9}$$

where $\bar{C}(B)$ is a constant depending only on B. For the third term in (4.4) we use the adjoint of (4.7)

$$P\left(E_{0k}^r\right) = P\left(E_{0k}^r\right)U_r\left(E_{0k}^r - H_0\right)^{-1} \tag{4.10}$$

to get

$$\left\|J_r P\left(E_{0k}^r\right)(1 - \tilde{J}_r)\right\| \leqslant \left\|U_r R_0\left(E_{0k}^r\right)(1 - \tilde{J}_r)\right\| \tag{4.11}$$

from which we obtain the same bound as in (4.9). Combining this result with (4.4), (4.5), (4.9) we obtain (4.1) in the proposition.

Let us now sketch the proof of (4.2). From the decoupling formula we have

$$\begin{aligned} R(z) - R_b(z) = {} & \left(\sum_{i\in\mathcal{I}} J_i R_i(z)\tilde{J}_i\right)\left(\sum_{n=1}^{\infty}\mathcal{K}(z)^n\right) - (1 - J_b)R_b(z) \\ & - J_b R_b(z)(1 - \tilde{J}_b) + J_\ell R_\ell(z)\tilde{J}_\ell + J_r R_r(z)\tilde{J}_r. \end{aligned} \tag{4.12}$$

Given an interval $\bar{\Delta} \subset \Delta_\varepsilon$ such that $\mathrm{dist}(\bar{\Delta}, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\varepsilon)}{2}L^{-p}$, we choose a circle $\bar{\Gamma}$ in the complex plane with diameter equal to $|\bar{\Delta}|$. Then if we integrate over $\bar{\Gamma}$ the last two terms on the right-hand side do not contribute while the second and third ones give $(1 - J_b)P_b(\bar{\Delta})$ and $J_b P_b(\bar{\Delta})(1 - J_b)$. Therefore

$$\begin{aligned} \|P - P_b(\bar{\Delta})\| \leqslant {} & |\bar{\Delta}|\left(\sum_{i\in\mathcal{I}}\sup_{z\in\bar{\Gamma}}\|R_i(z)\|\right)\frac{\sup_{z\in\bar{\Gamma}}\|\mathcal{K}(z)\|}{1 - \sup_{z\in\bar{\Gamma}}\|\mathcal{K}(z)\|} \\ & + \|(1 - J_b)P_b(\bar{\Delta})\| + \|J_b P_b(\bar{\Delta})(1 - \tilde{J}_b)\|. \end{aligned} \tag{4.13}$$

From lemma 1, $|\bar{\Delta}| < d(\varepsilon)L^{-1}$ and $\sup_{z\in\bar{\Gamma}}\|R_i(z)\| < \frac{2}{d(\varepsilon)}L^p$ the first term is bounded above by

$$12\varepsilon^{-1}C(B, V_0)L^p\,\mathrm{e}^{-\frac{B}{512}(\log L)^2}. \tag{4.14}$$

In order to estimate the second norm in (4.13) we note that (in the same way as in (4.6), (4.7))

$$P_b(\bar{\Delta}) = \sum_{E_\beta^b\in\bar{\Delta}} R_0\left(E_\beta^b\right)V_\omega P_b\left(E_\beta^b\right) \tag{4.15}$$

thus

$$\|(1 - J_b)P_b(\bar{\Delta})\| \leqslant \sum_{E_\beta^b\in\bar{\Delta}}\left\|(1 - J_b)R_0\left(E_\beta^b\right)V_\omega\right\|_2. \tag{4.16}$$

Each term of the sum can be bounded in a way similar to (3.10), and since the number of terms in the sum is equal to Tr $P_b(\bar{\Delta})$ we get

$$\begin{aligned} \|(1 - J_b)P_b(\bar{\Delta})\| & \leqslant \varepsilon^{-1}C(B, V_0)L\,\mathrm{e}^{-\frac{B}{64}(\log L)^2}\,\mathrm{Tr}\,P_b(\bar{\Delta}) \\ & \leqslant 2\varepsilon^{-3}c(B)^2C(B, V_0)V_0^2L^5\,\mathrm{e}^{-\frac{B}{64}(\log L)^2}. \end{aligned} \tag{4.17}$$

The second inequality follows from lemma 4 in appendix B (where we need $B > 4V_0$). For $\|J_b P_b(\bar{\Delta})(1 - \tilde{J}_b)\|$ one uses the adjoint of identity (4.15) to obtain the same result. The result (4.2) of the proposition then follows by combining (4.13), (4.14) and (4.17). □

In appendix B we adapt the method of [10] to our geometry to get the following Wegner estimate.

Proposition 2. *Let* $B \geqslant 4V_0$ *and* $E \in \Delta_\varepsilon$

$$\mathbb{P}_\Lambda\,(\mathrm{dist}(\sigma(H_b), E) < \delta) \leqslant 4c(B)\|h\|_\infty \delta\varepsilon^{-2} V_0 L^4. \tag{4.18}$$

Proof of theorem 1, parts (a) and (c). Let $\omega \in \Omega''_\Lambda$ where Ω''_Λ is the set given in proposition 1. Since for L large enough the right-hand side of (4.1) is strictly smaller than 1, the two projectors necessarily have the same dimension. Therefore, $\sigma(H_\omega) \cap I_k^\alpha$ contains a unique energy level E_k^α for each I_k^α of radius ρ. In particular, by taking the smallest value $\rho = \mathrm{e}^{-\frac{B}{512}(\log L)^2}$ we get (2.15). The number of such levels is $\mathcal{O}(L)$ since they are in one-to-one correspondence with the energy levels of H_α. The sets Σ_α of theorem 1 are precisely

$$\Sigma_\alpha = \bigcup_k \left(\sigma(H_\omega) \cap I_k^\alpha \cap \Delta_\varepsilon\right) \qquad \alpha = \ell, r. \tag{4.19}$$

The set of all other eigenvalues in $\sigma(H_\omega) \cap \Delta_\varepsilon$ defines Σ_b and is necessarily contained in intervals $\bar{\Delta}$ such that $\mathrm{dist}(\bar{\Delta}, \sigma(H_\ell) \cup \sigma(H_r)) = \frac{d(\varepsilon)}{2} L^{-p}$. In view of (2.15) this implies (2.17). Since the two projectors in (4.2) necessarily have the same dimension, the number of eigenstates in Σ_b is the same as that of $\sigma(H_b) \cap \Delta_\varepsilon$. It remains to estimate the probability of the set Ω''_Λ. The realizations of the complementary set are such that for at least one $E_{0k}^\alpha \in \Delta_\varepsilon$

$$\mathrm{dist}\left(\sigma(H_b), E_{0k}^\alpha\right) < d(\varepsilon) L^{-p} \tag{4.20}$$

but from proposition 2 this has a probability smaller than

$$4c(B)\|h\|_\infty d(\varepsilon) L^{-p} \varepsilon^{-2} V_0 L^4 \cdot \mathcal{O}(L) \tag{4.21}$$

where $\mathcal{O}(L)$ comes from the number of levels in $[\sigma(H_\ell) \cup \sigma(H_r)] \cap \Delta_\varepsilon$. Thus for L large enough

$$\mathbb{P}_\Lambda(\Omega''_\Lambda) \geqslant 1 - L^{6-p}. \tag{4.22}$$

We recall that $p \geqslant 7$. □

5. Estimates of currents

In this section we characterize the eigenvalues of H_ω in terms of the current carried by the corresponding eigenstates. This will yield parts (b) and (d) of theorem 2.

Proof of theorem 1, part (b). Let $E_k^\alpha \in \Sigma_\alpha$. By definition, the associated current is

$$J_k^\alpha = \mathrm{Tr}\, v_y P_k^\alpha \tag{5.1}$$

and will be compared to that of ψ_{0k}^α

$$J_{0k}^\alpha = \mathrm{Tr}\, v_y P\left(E_{0k}^\alpha\right). \tag{5.2}$$

The difference between these two currents will be estimated by $\left\|P_k^\alpha - P\left(E_{0k}^\alpha\right)\right\|$. First we observe that $v_y P_k^\alpha$ is trace class. Indeed, $v_y P_k^\alpha = v_y P_k^\alpha P_k^\alpha$ with $v_y P_k^\alpha$ bounded and $\left\|P_k^\alpha\right\|_1 = \mathrm{Tr}\, P_k^\alpha = 1$

$$\left\|v_y P_k^\alpha\right\|_1^2 \leqslant \left\|v_y P_k^\alpha\right\|^2 \leqslant \left\|P_k^\alpha v_y^2 P_k^\alpha\right\| \leqslant 2\left\|P_k^\alpha (H_\omega - V_\omega) P_k^\alpha\right\| \leqslant 2E_k^\alpha + V_0 \tag{5.3}$$

to get the second inequality one has simply added the positive terms to v_y^2. Similarly

$$\|v_y P(E_{0k}^\alpha)\|_1^2 \leqslant \|v_y P(E_{0k}^\alpha)\|^2 \leqslant \|P(E_{0k}^\alpha)\, v_y^2 P(E_{0k}^\alpha)\| \leqslant 2\,\|P(E_{0k}^\alpha)\, H_\alpha P(E_{0k}^\alpha)\| \leqslant 2E_{0k}^\alpha. \tag{5.4}$$

The identity

$$P_k^\alpha - P(E_{0k}^\alpha) = [P_k^\alpha - P(E_{0k}^\alpha)]^2 + [P_k^\alpha - P(E_{0k}^\alpha)]\, P(E_{0k}^\alpha) + P(E_{0k}^\alpha)\,[P_k^\alpha - P(E_{0k}^\alpha)] \tag{5.5}$$

implies

$$\left|J_k^\alpha - J_{0k}^\alpha\right| = \left|\mathrm{Tr}\, v_y\,[P_\alpha^k - P(E_{0k}^\alpha)]\right| \leqslant \left|\mathrm{Tr}\, v_y\,[P_k^\alpha - P(E_{0k}^\alpha)]^2\right| + \left|\mathrm{Tr}\, v_y\,[P_k^\alpha - P(E_{0k}^\alpha)]\, P(E_{0k}^\alpha)\right| + \left|\mathrm{Tr}\, v_y P(E_{0k}^\alpha)\,[P_k^\alpha - P(E_{0k}^\alpha)]\right|. \tag{5.6}$$

From (5.6), (5.3) and (5.4) we get

$$\left|J_k^\alpha - J_{0k}^\alpha\right| \leqslant 2\left(\|v_y P_k^\alpha\|_1 + \|v_y P(E_{0k}^\alpha)\|_1\right)\|P_k^\alpha - P(E_{0k}^\alpha)\| \leqslant 2((B+3V_0)^{1/2} + (B+2V_0)^{1/2})\,\|P_k^\alpha - P(E_{0k}^\alpha)\|. \tag{5.7}$$

Combining this last inequality with (4.1) we get the result (2.16) of theorem 1. □

In order to prove part (d) of theorem 1 we need the following lemma.

Lemma 2. *Fix $\omega \in \Omega'_\Lambda$ the set of realizations in $(H2)$. Let ψ_1^b, ψ_2^b be two eigenstates of H_b with eigenvalues E_1^b and E_2^b. Then*

$$\left|(\psi_1^b, v_y \psi_2^b)\right| \leqslant 2\left|E_1^b - E_2^b\right| L + \mathrm{e}^{-\frac{\mu(\varepsilon)}{4} L}. \tag{5.8}$$

For $\psi_1^b = \psi_2^b$, $E_1^b = E_2^b$, this shows that eigenstates of H_b do not carry any appreciable current. The main idea of the proof sketched below is that v_y is equal to the commutator $[-\mathrm{i}y, H_b]$ up to a small boundary term.

Proof. The wavefunctions ψ_1^b and ψ_2^b are defined on $\mathbb{R} \times \left[-\frac{L}{2}, \frac{L}{2}\right]$, are periodic along y and are twice differentiable in y. Here we will work with periodized versions of these functions where the y direction is infinite (but we keep the same notation). This allows us to shift integrals over y from $\left[-\frac{L}{2}, \frac{L}{2}\right]$ to $[\bar{y}_2, \bar{y}_2 + L]$. We have

$$(\psi_1^b, v_y \psi_2^b) = \int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y\, [\psi_1^b(\mathbf{x})]^* (-\mathrm{i}\partial_y - Bx)\psi_2^b(\mathbf{x}). \tag{5.9}$$

An integration by parts yields

$$\mathrm{i}\,(\psi_1^b, v_y \psi_2^b) = \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y\, [\psi_1^b(\mathbf{x})]^*\, y(-\mathrm{i}\partial_y - Bx)^2 \psi_2^b(\mathbf{x}) - \frac{1}{2}\int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y\, [(-\mathrm{i}\partial_y - Bx)^2 \psi_1^b(\mathbf{x})]^*\, y\psi_2^b(\mathbf{x}) + \mathcal{B} \tag{5.10}$$

where $\mathcal{B}$ is a boundary term given by

$$\mathcal{B} = \mathrm{i}\frac{L}{2}\int_{\mathbb{R}} \mathrm{d}x\, [(-\mathrm{i}\partial_y - Bx)\psi_1^b(x, \bar{y}_2)]^*\, \psi_2^b(x, \bar{y}_2) + [\psi_1^b(x, \bar{y}_2)]^* (-\mathrm{i}\partial_y - Bx)\psi_2^b(x, \bar{y}_2). \tag{5.11}$$

We can add a periodized version of V_ω and $\frac{1}{2}p_x^2$ to the kinetic energy operator in both terms on the right-hand side of (5.10) and use that ψ_1^b and ψ_2^b are eigenfunctions of H_b to obtain

$$\mathrm{i}\left(\psi_1^b, v_y\psi_2^b\right) = \left(E_2^b - E_1^b\right)\int_{\mathbb{R}} \mathrm{d}x \int_{\bar{y}_2}^{\bar{y}_2+L} \mathrm{d}y\, y\left[\psi_1^b(\mathbf{x})\right]^* \psi_2^b(\mathbf{x}) + \mathcal{B}. \tag{5.12}$$

From $|y| \leqslant |\bar{y}_2| + L \leqslant 2L$ and the Schwarz inequality we obtain

$$\left|\left(\psi_1^b, v_y\psi_2^b\right)\right| \leqslant 2L\left|E_2^b - E_1^b\right| + |\mathcal{B}|. \tag{5.13}$$

With the help of (C.6), (C.7) in appendix C we get

$$|\mathcal{B}| \leqslant \mathrm{e}^{-\frac{\mu(\varepsilon)}{4}L}. \tag{5.14}$$

This concludes the proof of (5.8). □

Proof of theorem 1, part (d). Let $\bar{\Delta}$ be an interval as in part (ii) of proposition 1. We consider the maximal set of intervals $\mathcal{F}_k \subset \bar{\Delta}$ such that $|\mathcal{F}_k| = \mathrm{e}^{-\frac{B}{1024}(\log L)^2}$ and $\mathrm{dist}(\mathcal{F}_k, \mathcal{F}_\lambda) \geqslant 4\,\mathrm{e}^{-\frac{B}{512}(\log L)^2}$, $k \neq \lambda$. Since the number of gaps between the $\mathcal{F}_k$ in $\bar{\Delta}$ is less than $\mathrm{e}^{\frac{B}{1024}(\log L)^2}|\bar{\Delta}|$ and $|\bar{\Delta}| < \frac{d(\varepsilon)}{L}$, it follows from proposition 2 that

$$\begin{aligned}\mathbb{P}_\Lambda(\Omega_\Lambda{}''') &\equiv \mathbb{P}_\Lambda\left(\omega \in \Omega_\Lambda : \sigma(H_b) \cap \bar{\Delta} \subset \bigcup_k \mathcal{F}_k\right)\\ &\geqslant 1 - 16c(B)\|h\|_\infty \varepsilon^{-2} V_0 L^4\, \mathrm{e}^{-\frac{B}{512}(\log L)^2}\, \mathrm{e}^{\frac{B}{1024}(\log L)^2}\frac{d(\varepsilon)}{L}\\ &= 1 - 16c(B)\|h\|_\infty \varepsilon^{-2} V_0 d(\varepsilon) L^3\, \mathrm{e}^{-\frac{B}{1024}(\log L)^2}.\end{aligned} \tag{5.15}$$

Now suppose that ψ_β is an eigenstate of H_ω corresponding to $E_\beta \in \bar{\Delta}$. For a given $\omega \in \Omega_\Lambda'''$ one can show that E_β is necessarily included in one of the fattened intervals $\tilde{\mathcal{F}}_k \equiv \mathcal{F}_k + \mathrm{e}^{-\frac{B}{512}(\log L)^2}$. In order to check this it is sufficient to adapt the estimates (4.13) to (4.17) to the difference of projectors $\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|$. The main point is to check that with our choice of intervals one is allowed to replace the circle $\bar{\Gamma}$ by circles $\bar{\Gamma}_k$ centred at the midpoint of $\mathcal{F}_k$ and of diameter $\mathrm{e}^{-\frac{B}{1024}(\log L)^2} + 2\,\mathrm{e}^{-\frac{B}{512}(\log L)^2}$. We do not give the details here. One finds

$$\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\| \leqslant \varepsilon^{-3} C''(B, V_0) L\, \mathrm{e}^{-\frac{B}{1024}(\log L)^2}. \tag{5.16}$$

Therefore, $P(\tilde{\mathcal{F}}_k)\psi_\beta = \psi_\beta$ for some k and we have

$$\begin{aligned}J_\beta = (\psi_\beta, v_y\psi_\beta) = (\psi_\beta, v_y P(\tilde{\mathcal{F}}_k)\psi_\beta) &= (P_b(\tilde{\mathcal{F}}_k)\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta) + ([P(\tilde{\mathcal{F}}_k)\\ &\quad - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta) + (\psi_\beta, v_y[P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta).\end{aligned} \tag{5.17}$$

To estimate the first term on the right-hand side of (5.17) we use the spectral decomposition in terms of eigenstates of H_b

$$P_b(\tilde{\mathcal{F}}_k)\psi_\beta = \sum_{E_\tau^b \in \tilde{\mathcal{F}}_k} \left(\psi_\tau^b, \psi_\beta\right)\psi_\tau^b. \tag{5.18}$$

We have

$$(P_b(\tilde{\mathcal{F}}_k)\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta) = \sum_{E_\tau^b, E_\sigma^b \in \tilde{\mathcal{F}}_k} \left(\psi_\beta, \psi_\tau^b\right)\left(\psi_\sigma^b, \psi_\beta\right)\left(\psi_\tau^b, v_y\psi_\sigma^b\right). \tag{5.19}$$

From lemmas 2 and 4 in appendix B we get

$$|(P_b(\tilde{\mathcal{F}}_k)\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta)| \leqslant (\mathrm{Tr}\, P_b(\mathcal{F}_k))^2 4L\, \mathrm{e}^{-\frac{B}{1024}(\log L)^2} \leqslant 16c(B)^4\varepsilon^{-4} V_0^4 L^9\, \mathrm{e}^{-\frac{B}{1024}(\log L)^2}. \tag{5.20}$$

The second term on the right-hand side of (5.17) is estimated by the Schwarz inequality

$$\begin{aligned}([P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta, v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta)^2 &\leqslant \|v_y P_b(\tilde{\mathcal{F}}_k)\psi_\beta\|^2 \|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ &\leqslant 2(P_b(\tilde{\mathcal{F}}_k)\psi_\beta, (H_b - V_\omega)P_b(\tilde{\mathcal{F}}_k)\psi_\beta)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ &\leqslant (B + 3V_0)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2. \end{aligned} \tag{5.21}$$

The third term is treated in a similar way

$$\begin{aligned}(\psi_\beta, v_y[P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)]\psi_\beta)^2 &\leqslant \|v_y\psi_\beta\|^2 \|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ &\leqslant 2(\psi_\beta, (H_\omega - V_\omega)\psi_\beta)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2 \\ &\leqslant (B + 3V_0)\|P(\tilde{\mathcal{F}}_k) - P_b(\tilde{\mathcal{F}}_k)\|^2. \end{aligned} \tag{5.22}$$

The last estimate (2.18) of theorem 1 then follows from (5.16), (5.20), (5.21) and (5.22). □

Remark. The set $\hat{\Omega}_\Lambda$ in theorem 1 may be taken equal to $\Omega'_\Lambda \cap \Omega_\Lambda{}'' \cap \Omega'''_\Lambda$. This set has a probability larger than $1 - 3L^{-s}$ with $s = \min(\theta, p - 6)$.

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Appendix A. Resolvent of the Landau Hamiltonian

The kernel $R_0(\mathbf{x}, \mathbf{x}'; z)$ of the resolvent $R_0(z) = (z - H_0)^{-1}$ with periodic boundary conditions along y can be expressed in terms of the kernel $R_0^\infty(\mathbf{x}, \mathbf{x}'; z)$ of the resolvent of the two-dimensional Landau Hamiltonian defined on the whole plane $\mathbb{R}^2$. Since the spectrum and the eigenfunctions of H_0 are exactly known, by writing down the spectral decomposition of $R_0(\mathbf{x}, \mathbf{x}'; z)$ and applying the Poisson summation formula we get for $z \in \rho(H_0)$

$$R_0(\mathbf{x}, \mathbf{x}'; z) = \sum_{m\in\mathbb{Z}} R_0^\infty(xy - mL, x'y'; z). \tag{A.1}$$

The formula for $R_0^\infty(\mathbf{x}, \mathbf{x}'; z)$ is (see for example [14])

$$R_0^\infty(\mathbf{x}, \mathbf{x}'; z) = \frac{B}{2\pi}\Gamma(\alpha_z)U\left(\alpha_z, 1; \frac{B}{2}|\mathbf{x} - \mathbf{x}'|^2\right) \mathrm{e}^{-\frac{B}{4}|\mathbf{x}-\mathbf{x}'|^2} M(\mathbf{x}, \mathbf{x}') \tag{A.2}$$

where $\alpha_z = \left(\frac{1}{2} - \frac{z}{B}\right)$ and

$$M(\mathbf{x}, \mathbf{x}') = \exp\left(\frac{\mathrm{i}}{2}B(x + x')(y - y')\right) \tag{A.3}$$

is the phase factor in the Landau gauge. In (A.2) the Landau levels appear as simple poles of the Euler Γ function and $U(-\lambda, b; \rho)$ is the logarithmic solution of the Kummer equation (see equations (13.1.1) and (13.1.6) of [37])

$$\rho\frac{\mathrm{d}^2U}{\mathrm{d}\rho^2} + (b - \rho)\frac{\mathrm{d}U}{\mathrm{d}\rho} + \lambda\rho = 0. \tag{A.4}$$

Lemma 3. *If* $|\mathrm{Im}\, z| \leqslant 1$, $\mathrm{Re}\, z \in \left]\frac{1}{2}B, \frac{3}{2}B\right[$ *and* $\frac{B}{2}|x - x'|^2 > 1$ *then, for L large enough, there exists* $C_n(z, B)$, $n = 0, 1$ *independent of L such that*

$$\left|\partial_x^n R_0(\mathbf{x}, \mathbf{x}'; z)\right| \leqslant C_n(z, B)\,\mathrm{e}^{-\frac{B}{8}(x-x')^2} \tag{A.5}$$

where $C_n(z, B) = C_n B^{1+\frac{n}{2}} \mathrm{dist}(z, \sigma(H_0))^{-1}$ *with* C_n *a numerical positive constant.*

For our purposes we need only decay in the x direction as provided by the lemma but in fact there is also a Gaussian decay in the y direction as long as $|y - y'| < \frac{L}{2}$. One can also prove similar estimates when $\mathrm{Re}\, z$ is between higher Landau levels but the constant is not uniform with respect to ν. Finally, we point out that this estimate does not hold for $\frac{B}{2}|\mathbf{x} - \mathbf{x}'|^2 < 1$ because of the logarithmic singularity in the Kummer function for $\rho \to 0$ (see also appendix C).

Proof. The proof relies on the estimate (6.10) of [14] which we state here for convenience. For $\lambda = x + \mathrm{i}y$, $N - 1 < x < N$ $(N \geqslant 1)$, $b \in \mathbb{N}$ and $\rho > 1$

$$|U(-\lambda, b; \rho)| \leqslant 2^{b+N-1}\rho^x(b + N + |y|)^N \frac{|\Gamma(-x)|}{|\Gamma(-\lambda)|} + \mathrm{e}^{-(\rho-2)}(\rho + 1 + |y|)^N \frac{(b + N)!}{|\Gamma(N - \lambda)|}. \tag{A.6}$$

Using this estimate for $N = 1$, $|y| < 1$ and $b = n$ together with $\Gamma(1 - \lambda) = -\lambda\Gamma(-\lambda)$ we have (C_n' a numerical constant)

$$|\Gamma(-\lambda)||U(-\lambda, n + 1; \rho)| \leqslant C_n'\rho\{\Gamma(-x) + |\lambda|^{-1}\}. \tag{A.7}$$

From (A.7) if $|\mathrm{Im}\, z| \leqslant 1$, $\mathrm{Re}\, z \in \left]\frac{1}{2}B, \frac{3}{2}B\right[$ and $\frac{B}{2}|\mathbf{x} - \mathbf{x}'|^2 > 1$ we deduce the estimate (C_n'' a numerical constant)

$$\left|\Gamma(\alpha_z)U\left(\alpha_z, n + 1; \frac{B}{2}|\mathbf{x} - \mathbf{x}'|^2\right)\right| \leqslant BC_n''\, \mathrm{dist}(z, \sigma(H_0))^{-1}|\mathbf{x} - \mathbf{x}'|^2. \tag{A.8}$$

From (A.8) for $n = 0$ and (A.1) we get

$$|R_0(\mathbf{x}, \mathbf{x}'; z)| \leqslant 2BC_0''\, \mathrm{dist}(z, \sigma(H_0))^{-1}\, \mathrm{e}^{-\frac{B}{8}(x-x')^2} \sum_{m\in\mathbb{Z}} \mathrm{e}^{-\frac{B}{8}(y-y'-mL)^2} \tag{A.9}$$

since $|y - y'| < L$ the last sum can be bounded by a constant, which yields (A.5) for $n = 0$.

To estimate the first derivative it is convenient to use the relation [37]

$$\frac{\mathrm{d}U(-\lambda, 1; \rho)}{\mathrm{d}\rho} = U(-\lambda, 1; \rho) - U(-\lambda, 2; \rho) \tag{A.10}$$

which yields

$$\begin{aligned} \partial_x R_0^\infty(\mathbf{x}, \mathbf{x}'; z) &= \frac{B}{2}[(x - x') + \mathrm{i}(y - y')]R_0^\infty(\mathbf{x}, \mathbf{x}'; z) \\ &\quad - B(x - x')\frac{B}{2\pi}\Gamma(\alpha_z)U\left(\alpha_z, 2; \frac{B}{2}|\mathbf{x} - \mathbf{x}'|^2\right) \mathrm{e}^{-\frac{B}{4}|\mathbf{x}-\mathbf{x}'|^2} M(\mathbf{x}, \mathbf{x}'). \end{aligned} \tag{A.11}$$

Using (A.8) to bound the two terms on the right-hand side of (A.11) we get

$$|\partial_x R_0^\infty(xy, x'y' - mL; z)| \leqslant B^{\frac{3}{2}} C_1''\, \mathrm{dist}(z, \sigma(H_0))^{-1}\, \mathrm{e}^{-\frac{B}{8}[(x-x')^2+(y-y'-mL)^2]} \tag{A.12}$$

the result (A.5) for $n = 1$ then follows from (A.12) and (A.1). □

Appendix B. Bounds on the number of eigenvalues in small intervals

We first prove a deterministic lemma on the maximal number of eigenvalues of H_b belonging to energy intervals I contained in Δ_ε. Then we sketch the proof of proposition 2. The ideas in this appendix come from the method used by Combes and Hislop to obtain the Wegner estimate which gives the expected number of eigenvalues in I. Since lemma 4 does not appear in [10] and we need to adapt the technique to our geometry we give some details below.

We begin with some preliminary observations on the kernel $P_0(\mathbf{x}, \mathbf{x}')$ of the projector onto the first Landau level with periodic boundary conditions along y. Using the spectral decomposition and the Poisson summation formula one gets

$$P_0(xy, x'y') = \sum_{m\in\mathbb{Z}} P_0^{\infty}(xy - mL, x'y') \tag{B.1}$$

where

$$P_0^{\infty}(\mathbf{x}, \mathbf{x}') = \frac{B}{2\pi}\, \mathrm{e}^{-\frac{B}{4}|\mathbf{x}-\mathbf{x}'|^2}\, \mathrm{e}^{\mathrm{i}\frac{B}{2}(x+x')(y-y')} \tag{B.2}$$

is the projector on the first Landau level for the infinite plane. The above formula can also be obtained by computing the residues of the poles of the Γ function. We observe that $V_{\mathbf{i}}^{1/2} P_0 V_{\mathbf{j}}^{1/2}$ is trace class. Indeed it is the product of two Hilbert–Schmidt operators $V_{\mathbf{i}}^{1/2} P_0$ and $P_0 V_{\mathbf{j}}^{1/2}$ and from the expression of the kernel (B.1) it is easily seen that ($c(B)$ a constant independent of L)

$$\left\| V_{\mathbf{i}}^{1/2} P_0 V_{\mathbf{j}}^{1/2} \right\|_1 \leqslant \left\| V_{\mathbf{i}}^{1/2} P_0 \right\|_2 \left\| P_0 V_{\mathbf{j}}^{1/2} \right\|_2 \leqslant c(B) V_0. \tag{B.3}$$

Lemma 4. *Let I be any interval contained in Δ_ε and $P_b(I)$ the eigenprojector associated with H_b. Then*

$$\operatorname{Tr} P_b(I) \leqslant 2\varepsilon^{-2} c(B)^2 V_0^2 L^4. \tag{B.4}$$

Proof. Let $Q_0 = 1 - P_0$ and E the middle point of I. Using $Q_0(H_0 - E)Q_0 \geqslant 0$ and $Q_0 R_0(E) Q_0 \leqslant (B - V_0)^{-1} Q_0$ we can write

$$\begin{aligned} P_b(I) Q_0 P_b(I) &= P_b(I) Q_0 (H_0 - E)^{1/2} R_0(E) (H_0 - E)^{1/2} Q_0 P_b(I) \\ &\leqslant (B - V_0)^{-1} P_b(I)(H_0 - E) Q_0 P_b(I) \\ &\leqslant (B - V_0)^{-1} \left[P_b(I)(H_b - E) Q_0 P_b(I) - P_b(I) V_\omega Q_0 P_b(I) \right] \end{aligned} \tag{B.5}$$

and thus from $\|P_b(I)(H_b - E)\| \leqslant \frac{|I|}{2}$, we get

$$\|P_b(I) Q_0 P_b(I)\| \leqslant (B - V_0)^{-1} \left(\frac{|I|}{2} + V_0 \right) \leqslant \frac{3V_0}{2(B - V_0)} \leqslant \frac{1}{2}. \tag{B.6}$$

In the last inequality we have assumed that $B \geqslant 4V_0$. Using $\operatorname{Tr} P_b(I) = \operatorname{Tr} P_b(I) P_0 P_b(I) + \operatorname{Tr} P_b(I) Q_0 P_b(I)$, $\operatorname{Tr} P_b(I) Q_0 P_b(I) \leqslant \|P_b(I) Q_0 P_b(I)\| \operatorname{Tr} P_b(I)$ and (B.6) we obtain

$$\operatorname{Tr} P_b(I) \leqslant 2 \operatorname{Tr} P_b(I) P_0 P_b(I) = 2 \operatorname{Tr} P_0 P_b(I) P_0. \tag{B.7}$$

Now, from

$$\operatorname{dist}\left(I, \frac{B}{2}\right)^2 P_b(I)^2 \leqslant \left(P_b(I) \left(H_b - \frac{B}{2} \right) P_b(I) \right)^2 \tag{B.8}$$

it follows that

$$\begin{aligned} \operatorname{Tr} P_0 P_b(I) P_0 &\leqslant \varepsilon^{-2} \operatorname{Tr} \left(P_0 P_b(I) \left(H_b - \frac{B}{2} \right) P_b(I) \left(H_b - \frac{B}{2} \right) P_b(I) P_0 \right) \\ &= \varepsilon^{-2} \operatorname{Tr}(P_0 V_\omega P_b(I) V_\omega P_0) \leqslant \varepsilon^{-2} \|P_0 V_\omega\|_2 \|V_\omega P_0\|_2 \end{aligned} \tag{B.9}$$

each Hilbert–Schmidt norm in (B.9) is bounded by $c(B) V_0 L^2$. This observation together with (B.7) gives the result of the lemma. □

Let us now sketch the proof of proposition 2.

Proof of proposition 2. Let $E \in \Delta_\varepsilon$ and $I = [E - \delta, E + \delta]$ for δ small enough (we require that I is contained in Δ_ε). By the Chebyshev inequality we have

$$\mathbb{P}_\Lambda\,(\mathrm{dist}(\sigma(H_b), E) < \delta) = \mathbb{P}_\Lambda\,(\mathrm{Tr}\, P_b(I) \geqslant 1) \leqslant \mathbb{E}_\Lambda(\mathrm{Tr}\, P_b(I)) \tag{B.10}$$

where $\mathbb{E}_\Lambda$ is the expectation with respect to the random variables in Λ. To estimate it we use an intermediate inequality of the previous proof

$$\mathbb{E}_\Lambda(\mathrm{Tr}\, P_b(I)) \leqslant 2\varepsilon^{-2}\mathbb{E}_\Lambda(\mathrm{Tr}\, P_0 V_\omega P_b(I) V_\omega P_0). \tag{B.11}$$

Writing $V_{\omega,\Lambda} = \sum_{\mathbf{i}\in\Lambda} X_{\mathbf{i}}(\omega) V_{\mathbf{i}}$

$$\begin{aligned}\mathrm{Tr}\, P_0 V_\omega P_b(I) V_\omega P_0 &= \sum_{\mathbf{i},\mathbf{j}\in\Lambda^2} X_{\mathbf{i}}(\omega) X_{\mathbf{j}}(\omega)\, \mathrm{Tr}\, P_0 V_{\mathbf{i}} P_b(I) V_{\mathbf{j}} P_0 \\ &= \sum_{\mathbf{i},\mathbf{j}\in\Lambda^2} X_{\mathbf{i}}(\omega) X_{\mathbf{j}}(\omega)\, \mathrm{Tr}\, V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2}.\end{aligned} \tag{B.12}$$

Since $V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2}$ is trace class we can introduce the singular value decomposition

$$V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} = \sum_{n=0}^{\infty} \mu_n(\psi_n, \cdot)\phi_n \tag{B.13}$$

where $\sum_{n=0}^{\infty} \mu_n = \left\| V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} \right\|_1$. Then

$$\begin{aligned}\mathrm{Tr}\, V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} &= \sum_{n=0}^{\infty} \mu_n \left(\phi_n, V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n\right) \\ &\leqslant \sum_{n=0}^{\infty} \mu_n \left(\phi_n, V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{i}}^{1/2} \phi_n\right)^{1/2} \left(\psi_n, V_{\mathbf{j}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n\right)^{1/2} \\ &\leqslant \frac{1}{2} \sum_{n=0}^{\infty} \mu_n \left\{ \left(\phi_n, V_{\mathbf{i}}^{1/2} P_b(I) V_{\mathbf{i}}^{1/2} \phi_n\right) + \left(\psi_n, V_{\mathbf{j}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n\right) \right\}.\end{aligned} \tag{B.14}$$

An application of the spectral averaging theorem of [10] shows that

$$\mathbb{E}_\Lambda\left(\left(\psi_n, V_{\mathbf{j}}^{1/2} P_b(I) V_{\mathbf{j}}^{1/2} \psi_n\right)\right) \leqslant \|h\|_\infty 2\delta \tag{B.15}$$

as well as for the term with $\mathbf{i}$ replacing $\mathbf{j}$ and ϕ_n replacing ψ_n. Combining (B.11), (B.14), (B.15) and (B.12) we get

$$\mathbb{E}_\Lambda(\mathrm{Tr}\, P_b(I)) \leqslant 4\|h\|_\infty \delta\varepsilon^{-2} \sum_{\mathbf{i},\mathbf{j}\in\Lambda^2} \left\| V_{\mathbf{j}}^{1/2} P_0 V_{\mathbf{i}}^{1/2} \right\|_1 \leqslant 4\|h\|_\infty \delta\varepsilon^{-2} c(B) V_0 L^4. \tag{B.16}$$

□

Appendix C. Estimate on the eigenfunction of H_b

In this section we prove Gaussian decay of the eigenfunction ψ_β^b and its y-derivative outside the support of the random potential V_ω. From the eigenvalue equation $(H_0 + V_\omega)\psi_\beta^b = E_\beta^b \psi_\beta^b$ we get

$$\psi_\beta^b = R_0\big(E_\beta^b\big)\, V_\omega \psi_\beta^b. \tag{C.1}$$

Thus

$$\begin{aligned}\left|\psi_\beta^b(\mathbf{x})\right| &\leqslant \int_{\mathbb{R}\times I_p} \left| R_0\big(\mathbf{x}, \mathbf{x}'; E_\beta^b\big)\, V_\omega(\mathbf{x}')\psi_\beta^b(\mathbf{x}')\right| \mathrm{d}\mathbf{x}' \\ &\leqslant V_0 \left\{ \int_{\mathrm{supp}\, V_\omega} \left| R_0\big(\mathbf{x}, \mathbf{x}'; E_\beta^b\big)\right|^2 \mathrm{d}\mathbf{x}' \right\}^{1/2}\end{aligned} \tag{C.2}$$

and

$$\left|\partial_y \psi^b_\beta(\mathbf{x})\right| \leqslant V_0 \sup_{\mathbf{x}} \left|\psi^b_\beta(\mathbf{x})\right| \int_{\operatorname{supp} V_\omega} \left|\partial_y R_0\left(\mathbf{x}, \mathbf{x}'; E^b_\beta\right)\right| \mathrm{d}\mathbf{x}'. \tag{C.3}$$

We need bounds on the integral kernel R_0 and its y-derivative to get an estimate of the eigenfunctions and their y-derivative. From [14] we have ($E \in \Delta_\varepsilon$)

$$\left|R_0^\infty(\mathbf{x}, \mathbf{x}'; E)\right| \leqslant C(B)|\Gamma(\alpha_E)|\, \mathrm{e}^{-\frac{B}{8}|\mathbf{x}-\mathbf{x}'|^2} \times \begin{cases} 1 & \text{if} \quad \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 > 1 \\ 1 + \left|\ln\left(\frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2\right)\right| & \text{if} \quad \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 \leqslant 1. \end{cases} \tag{C.4}$$

Calculating the y-derivative thanks to (A.10), and using bounds (6.16) of [14] we have

$$\left|\partial_y R_0^\infty(\mathbf{x}, \mathbf{x}'; E)\right| \leqslant C'(B)|\Gamma(\alpha_E)|\, \mathrm{e}^{-\frac{B}{8}|\mathbf{x}-\mathbf{x}'|^2} \times \begin{cases} 1 + |x| & \text{if} \quad \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 > 1 \\ \left(1 + \left|\ln\left(\frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2\right)\right|\right)(1 + |x| + |\mathbf{x}-\mathbf{x}'|^{-1}) & \text{if} \quad \frac{B}{2}|\mathbf{x}-\mathbf{x}'|^2 \leqslant 1. \end{cases} \tag{C.5}$$

With the help of (C.4) and (C.5) we can see that for L large enough

$$\left|\psi^b_\beta(\mathbf{x})\right| \leqslant C(B)\varepsilon^{-1} V_0 L \times \begin{cases} \mathrm{e}^{-\frac{B}{8}\left(x - \frac{L}{2} + \log L\right)^2} & \text{if} \quad x \notin \left[-\frac{L}{2}, \frac{L}{2}\right] \\ \ln(BL^2) & \text{if} \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right] \end{cases} \tag{C.6}$$

and

$$\left|\partial_y \psi^b_\beta(\mathbf{x})\right| \leqslant C'(B)\varepsilon^{-2} V_0^2 L^2 \times \begin{cases} \mathrm{e}^{-\frac{B}{8}\left(x - \frac{L}{2} + \log L\right)^2}(1 + |x|) & \text{if} \quad x \notin \left[-\frac{L}{2}, \frac{L}{2}\right] \\ L\left(\ln(BL^2)\right)^2(1 + |\mathrm{x}|) & \text{if} \quad x \in \left[-\frac{L}{2}, \frac{L}{2}\right] \end{cases} \tag{C.7}$$

Indeed, for $|m| > 1$, $\frac{B}{2}[(x - x')^2 + (y - y' - mL)^2] > 1$ thus we have

$$\left|R_0\left(\mathbf{x}, \mathbf{x}'; E^b_\beta\right)\right| \leqslant \tilde{C}(B)\varepsilon^{-1}\, \mathrm{e}^{-\frac{B}{8}(x-x')^2} + \sum_{|m|\leqslant 1} \left|R_0^\infty\left(xy, x'y' - mL; E^b_\beta\right)\right|. \tag{C.8}$$

If $x \notin \left[-\frac{L}{2}, \frac{L}{2}\right]$ since $\mathbf{x}' \in \operatorname{supp} V_\omega$ the terms $|m| \leqslant 1$ also have a Gaussian bound and

$$\left|R_0\left(\mathbf{x}, \mathbf{x}'; E^b_\beta\right)\right| \leqslant \tilde{C}'(B)\varepsilon^{-1}\, \mathrm{e}^{-\frac{B}{8}(x-x')^2}. \tag{C.9}$$

Replacing this bound in (C.2) we get the Gaussian decay in (C.6). On the other hand if $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$ we can use the logarithmic bounds for the terms $|m| \leqslant 1$ and we remark that they are integrable and bounded by $L^2 \ln(BL^2)$. The same arguments hold for the y-derivative.

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